An Explicit Parametrization of the Frenet Apparatus of the Slant Helix

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Abstract

Any two continuous functions κ and τ characterize a certain *Frenet curve* in 3-dimensional space. In principle, the curve can be constructed by solving the Frenet-Serret system of differential equations. Explicit solutions are known for *generalized helices*. In this paper, explicit characterizations and parametrizations are given for the Frenet apparatus of *slant helices*. In every point of a general helix, its tangent makes a constant angle with a fixed direction. Similarly, slant helices have a principal normal that has this property and their representations can be deduced by rearranging the Frenet apparatus of a helix. By the same method, a further class of curves can be constructed where the principal normal is the tangent of a slant helix, and so on.[1]

Keywords: Curve theory; Generalized helix; Slant helix; Frenet equations; Frenet curves

1 Frenet Space Curves

1.1 Regular Curves

This paper considers regular curves in oriented three-dimensional euclidian space $\mathbb{R}^3$, endowed with an *inner product*, $\langle u, v\rangle = \|u\|\|v\|\cos\theta$, and a *cross product*, $u \times v = \|u\|\|v\|\sin\theta N$, for any $u, v \in \mathbb{R}^3$, where $\theta \in [0, \pi]$ is the angle between u and v and N is the positively oriented unit vector perpendicular to the plane spanned by u and v. In particular, $u \perp v \Leftrightarrow \langle u, v\rangle = 0$ and u, v collinear $\Leftrightarrow u \times v = 0$.

A *parametrized curve* is a C^k map $x : I \mapsto \mathbb{R}^n$ (where $k \geq 0$ denotes the differentiation order) defined on an interval $I \subset \mathbb{R}$. We can picture it as the trajectory of a particle moving in

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[1] The results reported here were developed in 1996 as part of a master thesis

space, with position $x(t)$ and (if $k \geq 1$) velocity $\dot{x}(t) = \frac{dx}{dt}(t)$ at time t. The image $x[I]$ is called *trace* of the curve.

A parametrized curve x is called *regular* if it has a nonvanishing derivative $\dot{x}(t)$ at every point. A regular parametrized curve $x(t)$ has a well-defined unit tangent vector $T(t) = \dot{x}(t)/\|\dot{x}(t)\|$ and can be parametrized by arclength. Arclength will be denoted s throughout this paper and a prime denotes differentiation by arclength. The intrinsic geometric properties of regular curves are independent of regular parameter transformations and any regular curve can always be represented by an arclength (or *unit speed*) parametrization $x(s)$ with $x' = T$.

Let $x = x(s)$ be the arclength parametrization of regular C^2-curve and $T = x'$ its unit tangent vector. Then the continuous function

$$\kappa_+ = \|T'\| = \|x''\|$$

is called its *canonical curvature*. A point with $\kappa_+(s) = 0$ is called *inflection point*, otherwise *strongly regular*. A curve without inflection points is also called *strongly regular*. In strongly regular points, the *canonical principal normal vector* is defined as

$$N_+ = \kappa_+^{-1}\, T'.$$

Every regular point of a space curve in $\mathbb{R}^3$ has a well-defined *tangent line*, spanned by T, and perpendicular *normal plane*. Every strongly regular point has a well-defined *principal normal line* spanned by N_+, *osculating plane* spanned by T and N_+, and *rectifying plane* perpendicular N_+.

1.2 The Frenet Apparatus

Definition 1 (Frenet Curves and Frenet Apparatus)**.** Given a regular C^2 space curve with arclength parametrization $x : I \mapsto \mathbb{R}^3$ ($I \subset \mathbb{R}$ an interval) and unit tangent vector $T : I \mapsto S^2$, $T(s) = x'(s)$.

a) A differentiable unit vector field $N : I \mapsto S^2$ is a *principal normal* of the curve if in every point, $T \perp N$ and T' and N are collinear.

b) A differentiable unit vector field $B : I \mapsto S^2$ is a *binormal* of the curve if in every point, $T \perp B$ and $T' \perp B$.

c) A positively oriented orthonormal basis of differentiable vector fields (T, N, B) is called a *Frenet moving frame* of the curve if T is its tangent, N is a principal normal and B is a binormal.

d) Given a Frenet moving frame (T, N, B) of the curve, the following are well-defined and continuous:

   - The curvature $\quad \kappa : I \mapsto \mathbb{R}, \quad \kappa = \langle T', N \rangle$
   - The torsion $\quad \tau : I \mapsto \mathbb{R}, \quad \tau = \langle N', B \rangle$
   - The Lancret curvature $\quad \omega_+ : I \mapsto \mathbb{R}, \quad \omega_+ = \sqrt{\kappa^2 + \tau^2}$
   - The Darboux vector field $\quad D : I \mapsto \mathbb{R}^3, \quad D = \tau T + \kappa B$

The tuple $\mathcal{F} = (T, N, B, \kappa, \tau)$ is called a *Frenet apparatus* and the pair (κ, τ) a *Frenet development* associated with the curve.

e) A curve is called a *Frenet curve* if it has a Frenet moving frame.

Remarks. 1. Given a tangent and either a principal normal or a binormal, the third component of the moving frame can be constructed as $B = T \times N$ or $N = B \times T$, respectively. The existence of either a principal normal or a binormal is therefore sufficient for the existence of a Frenet moving frame and apparatus.

2. By definition, a Frenet curve is regular and at least of order C^2. The components of the Frenet frame are at least of order C^1 and the Frenet development at least of order C^0.

3. Every strongly regular C^3-curve is a Frenet curve. It can be assigned a uniquely defined, canonical Frenet apparatus with principal normal $N = N_+$ and curvature $\kappa = \kappa_+ > 0$.

4. Whether or not a regular but not strongly regular curve is a Frenet curve depends on whether the map mapping each strongly regular point to the osculating plane has a continuous extension. Curves that discontinuously "jump" between planes cannot be Frenet-framed.

5. Frenet curves with inflection points cannot be assigned a canonical Frenet apparatus. If (T, N, B, κ, τ) is a Frenet apparatus of a Frenet curve, then so is $(T, -N, -B, -\kappa, +\tau)$. In strongly regular points, $N = \pm N_+$ and $\kappa = \pm\kappa_+$. It follows that if a Frenet curve has no or only isolated inflection points, then its Frenet apparatus is unique up to the sign of N, B, and κ and in particular, it has a unique and well-defined torsion τ.

6. A Frenet curve containing straight line segments has an infinite family of Frenet apparatuses because on a straight line, the curvature always vanishes ($T' = 0$) but the torsion can vary arbitrarily. In general (Wong and Lai (1967)), two Frenet developments (κ, τ) und $(\tilde{\kappa}, \tilde{\tau})$ characterize the same Frenet curve if and only if there is a a C^1-Funktion θ so that

$$\kappa \sin\theta = 0, \qquad \kappa\cos\theta = \tilde{\kappa} \qquad \text{and} \quad \theta' = \tilde{\tau} - \tau.$$

7. The definition of Frenet curve presented here (cf. Wintner (1956), Wong and Lai (1967)) is justified by the fact that important classes of curves can be treated as Frenet curves even if they have inflection points. Examples are regular plane curves, generalized helices and slant helices (see section 2 of the present paper) as well as curves that are *geodesics* or *asymptotic curves* on a regular surface. In the case of asymptotic curves (this includes plane curves), the surface normal in every point of the curve can be chosen as binormal. In the case of geodesics, the surface normal can be chosen as principal normal. The prize to be paid for relaxing the assumption of strong regularity and $\kappa > 0$ is the loss of uniqueness of the Frenet apparatus.

Theorem 1 (Frenet Equations)**.** Any Frenet apparatus (T, N, B, κ, τ) satisfies the *Frenet equations*

$$T' = \kappa N, \qquad N' = -\kappa T + \tau B, \qquad B' = -\tau N.$$

In matrix notation, we have

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \cdot \begin{pmatrix} T \\ N \\ B \end{pmatrix} = D \times \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Proof. For any two differentiable unit vector fields V and W keeping a constant angle, it is

$$\langle V, W\rangle' = 0 \Rightarrow \langle V', W\rangle + \langle V, W'\rangle = 0 \Rightarrow \langle V', W\rangle = -\langle V, W'\rangle.$$

For any C^1 unit vector field V, $V' \perp V$. It follows that differentiating a moving orthonormal basis gives rise to a skew symmetric matrix of coefficient functions. By definition 1, they are $\kappa = \langle T', N\rangle = -\langle T, N'\rangle$ and $\tau = \langle N', B\rangle = -\langle N, B'\rangle$, and $\langle T', B\rangle = \langle B', T\rangle = 0$. □

The right hand side of the Frenet equation suggests that the *Darboux rotation vector* D can be interpreted as the angular momentum vector of the Frenet moving frame. Its direction determines the moving frame's momentary axis of motion (its *centrode*) and its length the angular speed $\|D\| = \omega_+$. We also note the relationships

$$\kappa^2 = \kappa_+^2 = \langle T', T'\rangle, \qquad \omega_+^2 = \langle N', N'\rangle, \qquad \tau^2 = \langle B', B'\rangle.$$

Given a pair of continuous coefficient functions $\kappa = \kappa(s)$ and $\tau = \tau(s)$ - also known as the *natural equations* of a curve - and a set of initial conditions, the Frenet system of linear differential equations is guaranteed to have a unique solution. A Frenet development therefore determines a Frenet moving frame (unique up to a rotation) which is associated with a unique (up to a translation) Frenet curve, an arclength parametrization of which can be constructed by integration over the tangent vector contained in the moving frame. Strongly regular C^3-curves have a canonical Frenet apparatus and are completely (up to a rigid motion) characterized by their canonical curvature and torsion. Two strongly regular space curves are congruent if and only if their curvature and torsion are identical (in arclength parametrization). This fact is known as the *fundamental theorem of space curves* and it is usually stated under the assumption that $\kappa > 0$ and differentiable (e. g. Kühnel (2006)). When the assumption of strong regularity is relaxed, the fundamental theorem still essentially holds except that there is no unique, canonical curvature. Two different Frenet developments may now characterize the same curve (see remark 6).

An important question in curve theory is whether explicit solutions to the Frenet equations can be found given κ and τ. The following theorem describes a method for transforming a Frenet apparatus of a curve to obtain the Frenet apparatus of a related curve. It is useful for finding parametrizations of helices and slant helices.

Theorem 2 (Transformation of Frenet Apparatus)**.** Given a Frenet curve $x(s)$ with Frenet apparatus $\mathcal{F} = (T, N, B, \kappa, \tau)$ and the transformation

$$\begin{aligned}
T_1 &= -\sin\varphi N + \cos\varphi B \\
N_1 &= T \\
B_1 &= \cos\varphi N + \sin\varphi B \\
\kappa_1 &= \kappa \sin\varphi \\
\tau_1 &= \kappa \cos\varphi \\
\varphi(s) &= \varphi_0 - \int_{s_0}^{s} \tau(\sigma) d\sigma
\end{aligned}$$

with some constant $\varphi_0 \in \mathbb{R}$. Then $\mathcal{F}_1 = (T_1, N_1, B_1, \kappa_1, \tau_1)$ is also a Frenet apparatus of a Frenet curve x_1. Its Darboux vector is $D_1 = \kappa B$.

Proof. We need to show (T_1, N_1, B_1) is a positively oriented moving frame of orthogonal unit vectors satisfying the Frenet equations with curvature κ_1 and torsion τ_1. Let

$$A = \begin{pmatrix} 0 & -\sin\varphi & \cos\varphi \\ 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \end{pmatrix}, \; F = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}, \; \mathcal{B} = \begin{pmatrix} T \\ N \\ B \end{pmatrix} \text{ and } \mathcal{B}_1 = \begin{pmatrix} T_1 \\ N_1 \\ B_1 \end{pmatrix}.$$

A is orthogonal with $\det A = +1$, therefore $A^{-1} = A^t$, and we have $\mathcal{B}_1 = A\mathcal{B}$, $\mathcal{B}' = F\mathcal{B}$, and $\mathcal{B}_1' = A'\mathcal{B} + A\mathcal{B}' = (A' + AF)\mathcal{B} = (A' + AF)A^t\mathcal{B}_1$. Denoting $F_1 = A'A^t + AFA^t$, calculation confirms that

$$F_1 = \begin{pmatrix} 0 & \kappa\sin\varphi & 0 \\ -\kappa\sin\varphi & 0 & \kappa\cos\varphi \\ 0 & -\kappa\cos\varphi & 0 \end{pmatrix}.$$

By construction, therefore, (T_1, N_1, B_1) is a positively oriented moving frame satisfying the Frenet equations $\mathcal{B}_1' = F_1\mathcal{B}_1$. □

Remarks. 1. This transformation (considered in similar form in Bilinski (1963)) can be derived from converting a curve's Frenet frame to its *Bishop frame* (which consists of tangent and two normal vectors with tangential derivatives, see Bishop (1975) for details), then rearranging it into a new Frenet frame. x_1 has a Frenet apparatus with the tangent of x as its principal normal. Moreover, its Darboux vector is $D_1 = \kappa B$ so the Frenet frame of x_1 is momentarily rotating around the binormal of x at angular speed $\omega_1 = |\kappa|$.

2. If is interesting to consider the case when the Frenet apparatus is periodic, $\mathcal{F}(s+L) = \mathcal{F}(s)$ for all s, which is necessary but not sufficient for the curve to be closed. The transformed apparatus $\mathcal{F}_1$ is also periodic if and only if the *total torsion* of $\mathcal{F}$, defined as $\int_0^L \tau(s)ds$, is a rational multiple of 2π.

2 Helices and Slant Helices

2.1 Plane Curves

Definition 2 (Constant Slope, Fixed Normal)**.** A vector field $v : I \mapsto R^n$ is said to have *constant slope* if it makes a constant angle $\theta \in (0, \pi/2]$ with a fixed direction, represented by a unit vector V_0 called *slope vector*. If the angle is $\pi/2$, then V_0 is a *fixed normal* of the vector field.

Remarks. A unit vector field V has constant slope θ if and only if there is a unit vector V_0 so that $\langle V, V_0\rangle \equiv \cos\theta$. Obviously, a unit vector field with constant slope traces part of a circle on the unit sphere, and if $\theta = \pi/2$, it traces part of a great circle. A vector field with a fixed normal traces a plane curve.

Lemma 1. A differentiable unit vector field $V : I \mapsto R^n$ has constant slope if and only if the slope vector is a fixed normal of V'.

Proof. V has constant slope with $V_0 \Leftrightarrow \langle V, V_0\rangle = const. \Leftrightarrow \langle V, V_0\rangle' = \langle V', V_0\rangle + \langle V, V_0'\rangle = 0 \Leftrightarrow \langle V', V_0\rangle = 0.$ □

Theorem 3 (Plane Curves)**.** For a regular C^2 space curve $x(s)$ with unit tangent vector T, the following conditions are equivalent:

i.) x lies on a plane.

ii.) T has a fixed normal.

iii.) x has a constant binormal.

iv.) x has a Frenet apparatus with vanishing torsion.

Given continuous functions $\kappa = \kappa(s)$ and $\tau \equiv 0$ defined on an interval $s_0 \in I \subset \mathbb{R}$. Let $\Omega(s) = \int_{s_0}^{s} \kappa(\sigma) d\sigma$ and

$$T_P(s) = \begin{pmatrix} -\sin \Omega(s) \\ \cos \Omega(s) \\ 0 \end{pmatrix}, \quad N_P(s) = \begin{pmatrix} -\cos \Omega(s) \\ -\sin \Omega(s) \\ 0 \end{pmatrix}, \quad B_P = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then $\mathcal{F}_P = (T_P, N_P, B_P, \kappa, \tau)$ is a Frenet apparatus of a plane curve. Its tangent and principal normal trace parts of great circles, its binormal is constant and its axis of motion is fixed, with Darboux vector $D_P = \kappa B_P$.

Proof. (i) $\Rightarrow$ (ii): Let $x(s)$ be a regular plane curve with plane normal B_0. Then if $x(s_0)$ is a curve point, B_0 is a fixed normal of the vector field $x(s) - x(s_0) \Rightarrow \langle x'(s), B_0 \rangle = \langle T(s), B_0 \rangle = 0$.

(ii) $\Rightarrow$ (iii): T has a fixed normal B_0 with $T \perp B_0 \Rightarrow \langle T'(s), B_0 \rangle = 0$ (lemma 1). By definition 1, B_0 is a binormal.

(iii) $\Rightarrow$ (iv): Given a constant binormal B_0, $\tau^2 = \langle B_0', B_0' \rangle \equiv 0$.

(iv) $\Rightarrow$ (i): If a Frenet development $\kappa = \kappa(s)$ and $\tau \equiv 0$ is given, the binormal must be constant, $B = B_0$. $\langle x(s) - x(s_0), B_0 \rangle' = \langle x'(s), B_0 \rangle + \langle x(s) - x(s_0), B_0' \rangle = 0$. It follows that the curve must lie on a plane.

Finally, it is easy to verify that $\mathcal{F}_P$ satisfies the definition of a Frenet apparatus. $\square$

2.2 Generalized Helices

Definition 3 (Generalized Helix, Slope Line)**.** A regular C^2-curve is called a *generalized helix* (also *slope line* or *curve of constant slope*) if at every point it makes a constant angle $\theta \in (0, \pi/2)$ with a fixed direction.

Remark. The excluded cases correspond to straight lines ($\theta = 0$) and plane curves ($\theta = \pi/2$).

Theorem 4 (Generalized Helices)**.** For a regular C^2 space curve x with unit tangent vector T, the following conditions are equivalent:

i.) x is a generalized helix.

ii.) T has constant slope θ.

iii.) x has a principal normal that has a fixed normal.

iv.) x is a Frenet curve and has a Frenet apparatus with $\tau = \cot\theta\kappa$.

Proof. (i) $\Leftrightarrow$ (ii): By definition.

(ii) $\Leftrightarrow$ (iii): Let D_0 be the slope vector, then $N = \frac{1}{\sin\theta} D_0 \times T$ is a unit vector with fixed normal D_0. $T' \perp D_0$ (lemma 1) and $T' \perp T$ imply that T' is collinear N, so N is a principal normal. Conversely, if N is a principal normal collinear T', and $N \perp D_0$, then is also $T' \perp D_0$ and by lemma 1, T has constant slope.

(ii) $\Rightarrow$ (iv): Let D_0 be the slope vector. Define $B = \csc\theta D_0 - \cot\theta T$ so that $D_0 = \cos\theta T + \sin\theta B$. By lemma 1, $T' \perp D_0$. It is also $T' \perp T$ so $T' \perp B$. By construction, $T \perp B$. By definition 1, B is a binormal. By construction, B has constant slope $\pi/2 - \theta$ with D_0. Differentiating B gives $B' = -\cot\theta T' = -\cot\theta\kappa N \Rightarrow \tau = \cot\theta\kappa$.

(iv) $\Rightarrow$ (ii): We can construct the vector $D_0 = \cos\theta T + \sin\theta B$. $D_0' = \cos\theta\kappa N - \sin\theta\tau N = 0 \Rightarrow D_0 = const$. By construction, T has slope θ and B has slope $\pi/2 - \theta$ with D_0.

The slope vector D_0 is the direction of the axis of rotation:

$$D = \tau T + \kappa B = \kappa/\sin\theta D_0, \ \omega = \csc\theta\kappa$$

$\square$

Theorem 5 (Frenet Apparatus of Generalized Helix)**.** Given a constant $\theta \in (0, \pi/2)$ and continuous functions κ, $\kappa_H = \sin\theta\,\kappa$, $\tau_H = \cot\theta\,\kappa_H = \cos\theta\,\kappa : I \mapsto \mathbb{R}$, $s_0 \in I$. Let $\Omega(s) = \int_{s_0}^{s} \kappa(\sigma) d\sigma$ and

$$T_H(s) = \begin{pmatrix} \sin\theta\cos\Omega(s) \\ \sin\theta\sin\Omega(s) \\ \cos\theta \end{pmatrix}, \ N_H(s) = \begin{pmatrix} -\sin\Omega(s) \\ \cos\Omega(s) \\ 0 \end{pmatrix}, \ B_H(s) = \begin{pmatrix} -\cos\theta\cos\Omega(s) \\ -\cos\theta\sin\Omega(s) \\ \sin\theta \end{pmatrix}.$$

Then $\mathcal{F}_H = (T_H, N_H, B_H, \kappa_H, \tau_H)$ is a Frenet apparatus of a generalized helix with slope angle θ. Its tangent has constant slope θ, its binormal has constant slope $\pi/2 - \theta$ and its principal normal lies on a great circle. Its axis of motion is fixed in the direction of the slope vector, with angular speed κ.

Proof. This is a direct application of theorem 2 to the Frenet apparatus of a plane curve (theorem 3) with $\varphi = \theta = const.$ □

2.3 Slant Helices

Definition 4 (Slant Helix)**.** A Frenet curve is called a *slant helix* if it has a principal normal that has constant slope (Izumiya and Takeuchi (2004)).

A principal normal vector of constant slope is exactly a tangent vector of a generalized helix. Applying the transformation of theorem 2 to the Frenet apparatus of a generalized helix gives a Frenet apparatus of a slant helix.

Theorem 6 (Frenet Apparatus of Slant Helix)**.** *A Frenet curve is a slant helix if and only if it has a Frenet development*

$$\kappa_{SH} = \omega \sin\varphi, \qquad \tau_{SH} = \omega\cos\varphi, \qquad \varphi(s) = \varphi_0 - \cot\theta \int_{s_0}^{s} \omega(\sigma)d\sigma$$

($\omega : I \mapsto \mathbb{R}$ a continuous function, $s_0 \in I \subset \mathbb{R}$, $\varphi_0 \in \mathbb{R}$, $\theta \in (0, \pi/2)$).

Given such a Frenet development and let $\lambda_1 := 1 - \cos\theta, \lambda_2 := 1 + \cos\theta$. Then the tangent vector of the slant helix thus characterized can be parametrized as follows:

$$T_{SH} = \frac{1}{2}\begin{pmatrix} \lambda_1 \cos\lambda_2\Omega - \lambda_2\cos\lambda_1\Omega \\ \lambda_1\sin\lambda_2\Omega - \lambda_2\sin\lambda_1\Omega \\ 2\sin\theta\cos(\cos\theta\cdot\Omega)\end{pmatrix}, \ \Omega(s) = \frac{\varphi_0}{\cos\theta} + \frac{1}{\sin\theta}\int_{s_0}^{s}\omega(\sigma)d\sigma.$$

The regular arcs traced by T are spherical helices and the slant helix has a Darboux vector with constant slope $\pi/2 - \theta$.

Proof. Assume that a curve is a slant helix. It has a principal normal N making constant slope θ with some unit vector D_0. We can write $D_0 = \cos\theta N + \sin\theta(\cos\varphi T + \sin\varphi B)$ with some differentiable function φ. We then have $D_0' = -\cos\theta\kappa T + \cos\theta\tau B + \varphi'\sin\theta(-\sin\varphi T + \cos\varphi B) + \sin\theta(\cos\varphi\kappa N - \sin\varphi\tau N)$. From $D_0' = 0$ derive three conditions:

$$\kappa = -\tan\theta\,\varphi'\sin\varphi, \quad \tau = -\tan\theta\,\varphi'\cos\varphi, \quad \cos\varphi\;\kappa = \sin\varphi\;\tau.$$

Setting $\omega = -\tan\theta\,\varphi'$, we get $\kappa = \omega\sin\varphi$, $\tau = \omega\cos\varphi$, and $\varphi' = -\cot\theta\,\omega$ as claimed.

Now the transformation described in theorem 2 is applied to the Frenet apparatus of the generalized helix (theorem 4), with $\omega = \kappa_H$. The result is a Frenet apparatus of a slant helix, because the tangent of the helix becomes the principal normal of the slant helix, and its Frenet development is exactly as stated in the premise. The calculation of T_{SH} is shown; parametrizations of principal normal and binormal are obtained in the same way but are omitted here.

By theorem 2, we have $T_{SH} = -\sin\varphi N_H + \cos\varphi B_H$, $\varphi = \varphi_0 - \cot\theta \int \omega$. Therefore

$$T_{SH} = \begin{pmatrix} +\sin\varphi \sin\Omega - \cos\theta \cos\varphi \cos\Omega \\ -\sin\varphi \cos\Omega - \cos\theta \cos\varphi \sin\Omega \\ \sin\theta \cos\varphi \end{pmatrix}, \Omega = \Omega_0 + \frac{1}{\sin\theta} \int \omega.$$

The integration constant Ω_0 can be set arbitrarily because it only rotates the moving frame. By setting $\Omega_0 = -\varphi_0 / \cos\theta$, we get $\varphi = -\cos\theta\Omega$ and evaluation yields the expression above.

The regular arcs of the curve described by the unit tangent T are slope lines because $T' = \kappa N$ has constant slope. By construction, the unit Darboux vector of the slant helix is the binormal of a helix, which has constant slope (see theorem 2). □

Remarks. 1. The natural equations for slant helices derived above are of unrestricted generality. Any choice of a continuous function $\omega(s)$ and constants θ and φ_0 gives rise to a well-defined slant helix and its Frenet apparatus, with Frenet development

$$\kappa_{SH} = \omega \sin\varphi = -\tan\theta\, \varphi' \sin\varphi, \qquad \tau_{SH} = \omega \cos\varphi = -\tan\theta\, \varphi' \cos\varphi, \qquad \varphi' = -\cot\theta\, \omega.$$

Inflection points and even straight line segments are not excluded. The function $\omega(s)$ is the signed Lancret curvature of the curve and has a geometric interpretation as the Frenet frame's speed of precession.

2. It is easy to verify that under the assumption $\kappa \neq 0$, the natural equations are equivalent to the condition

$$\frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)' = \cot\theta = const.$$

This expression is equivalent to the *geodesic curvature* of the spherical image of the principal normal (see Izumiya and Takeuchi (2004)), which is part of a circle.

3. The differential equation $-\varphi' \sin\varphi = \cot\theta\kappa$ can be solved for φ. Setting $m = \cot\theta$, we get $-\sin\varphi\, d\varphi = m\, \kappa\, ds \;\Rightarrow\; \cos\varphi(s) = m \int \kappa\, ds$. Further, we have $\tau = \kappa \cot\varphi = \kappa \frac{\cos\varphi}{\sqrt{1-\cos^2\varphi}}$ (here we must assume $\varphi \in (0, \pi)$). It follows

$$\tau(s) = \kappa(s) \frac{m \int \kappa(s) ds}{\sqrt{1 - m^2 (\int \kappa(s) ds)^2}}.$$

This condition is due to Ali (2012). Given an arbitrary curvature κ, a torsion function can be constructed to create a slant helix, provided that the domain of s is approporiately restricted. One application is the construction of a *Salkowski curve* (for details see Ali (2012)) with

$$\kappa(s) \equiv 1, \quad \tau(s) = \frac{ms}{\sqrt{1 - m^2 s^2}}, \quad s \in (-1/m, +1/m).$$

4. From $\cos\varphi(s) = m \int \kappa\, ds$ and similar $\sin\varphi = -m \int \kappa\, ds$ we can deduce formulae for the total curvature and total torsion over a curve segment of length l:

$$\int_0^l \kappa(s)\, ds = \frac{1}{m}(\cos\varphi(l) - 1), \qquad \int_0^l \tau(\sigma)\, d\sigma = -\frac{1}{m} \sin\varphi(l).$$

For a closed Frenet curve, total curvature (up to sign) and torsion are independent of the choice of the Frenet apparatus even when inflection points are present. In the case of a closed slant helix with length L, with both ω and $\varphi \mod 2\pi$ periodic with L, both total curvature and torsion vanish. Total curvature has a geometric interpretation as follows. On each inflection point free curve segment, the absolute total curvature is equivalent to the length of the arc traced by the tangent on the sphere (Scofield (1995)). Inflection points appear as cusps in the tangent image. Vanishing total curvature means that the arc lengths of the tangent image between the cusps add up to zero, when each arc "length" is assigned the sign of the curvature. An example follows.

5. The case $\omega = const.$ gives rise to the class of *curves of constant precession* (Scofield (1995)). Setting $\mu = \cot\theta\ \omega$ and $\alpha = \sqrt{\omega^2 + \mu^2}$, and omitting φ_0 as it only effects a phase shift, we can substitute $\cos\theta = \mu/\alpha, \sin\theta = \omega/\alpha,\ \lambda_1 = (\alpha - \mu)/\alpha,\ \lambda_2 = (\alpha + \mu)/\alpha,\ \Omega = \alpha s$ in the formulae for the slant helix.

Theorem 7 (Curves of Constant Precession)**.** A *curve of constant precession* has curvature, torsion and tangent parametrization as follows :

$$\kappa_{CP} = -\omega \sin \mu s,\ \tau_{CP} = \omega \cos \mu s \quad (\omega, \mu \neq 0 \text{ arbitrary constants}),$$

$$T_{CP} = \frac{1}{2\alpha} \begin{pmatrix} (\alpha - \mu)\cos(\alpha + \mu)s - (\alpha + \mu)\cos(\alpha - \mu)s \\ (\alpha - \mu)\sin(\alpha + \mu)s - (\alpha + \mu)\sin(\alpha - \mu)s \\ 2\omega \cos \mu s \end{pmatrix}.$$

An explicit arclength parametrization of the curve is obtained by elementary integration. The resulting curve of constant precession lies on a hyperboloid of one sheet, and it is closed if and only if μ/α is rational (figure 2 in Scofield (1995)).

Remark. Curves of constant precession arise by transformation of the Frenet apparatus of a circular helix with curvature ω and torsion μ, which is periodic with period $L = 2\pi/\alpha$ (although the helix itself is not closed). The total torsion of the helix is $2\pi\mu/\alpha$ and so the transformed Frenet apparatus is periodic if and only if if μ/α is rational. In this case, we obtain a closed curve. Under what conditions a periodic Frenet apparatus is associated with a closed curve is an open problem, known as the *closed curve problem*. Interestingly, in accordance with remark 4 above, both the total curvature and total torsion of a closed curve of constant precession vanish. This implies that each "upward" arc of the tangent image is balanced out by a "downward" arc of the same length (figure 1 in Scofield (1995)).

Outlook

In this paper, a certain transformation was applied first to a the Frenet apparatus of a generic plane curve to obtain a generic helix, and then again to obtain a slant helix. This procedure could be repeated ad infinitum to give rise to new, yet unexplored classes of curves. In particular, applying the transformation to a closed curve of constant precession shoud result in a curve with periodic Frenet apparatus and potentially also closed.

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